

FIXED POINT THEOREMS FOR GROUPS ACTING ON NON-POSITIVELY CURVED MANIFOLDS

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ABSTRACT. We study isometric actions of Steinberg groups on Hadamard manifolds. We prove some rigidity properties related to these actions. In Particular we show that every isometric action of $St_n(\mathbb{F}_p\langle t_1, \dots, t_k \rangle)$ on Hadamard manifold when $n \geq 3$ factors through a finite quotient.. We further study actions on infinite dimensional manifolds and prove a fixed point theorem related to such actions.

1. INTRODUCTION

We study isometric actions of non-commutative Steinberg groups on Hadamard manifolds. Hadamard manifolds are complete simply connected non-positively curved Riemannian manifolds. Usually Hadamard manifolds are assumed to be of finite dimension. We also consider the infinite dimension case. Recall that while finite dimensional manifolds are metrically proper (i.e. closed balls are compact), infinite dimensional manifolds are not hence we will have different treatment for each case.

It is a well known question of Gromov whether there exist groups with no fixed point free action on CAT(0) spaces. Gromov conjectured that random groups have this property (see Pansu [17]). A first step in this direction was done by Arzhantseva et al.. They introduced an example of infinite group that admits no non-trivial isometric action on finite dimensional manifolds which are p-acyclic [1]. Next it was shown by Naor and Silberman [16] that indeed not only that random groups have fixed points when acting on CAT(0) spaces, but that this property can be extended to many p-convex metric spaces.

We focus our attention on the higher rank Steinberg groups, $St_n(R)$ when $n \geq 3$ and R is either the associative ring $R = \mathbb{F}_p\langle t_1, \dots, t_k \rangle$ (for some applications we require that $p \geq 5$) or the torsion free ring $R = \mathbb{Z}\langle t_1, \dots, t_k \rangle$ (we use the $\langle \rangle$ sign to denote non-commutative polynomials). These groups are often denoted as non-commutative universal lattices. Kassabov (and Shalom in the commutative case) coined the name as they surject on many lattices in higher rank Lie groups. It is for this reason that any fixed point property proved for them immediately applies for the corresponding lattices. Since lattices in p-adic analytic groups and in Lie groups do have fixed point free actions on CAT(0) spaces (their associated buildings and symmetric spaces for example) one can not hope to have such a strong result concerning their actions. We have therefore to assume more.

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Our first goal is to study isometric actions of $St_n(\mathbb{F}_p\langle t_1, \dots, t_k \rangle)$ on (finite dimensional) Hadamard manifolds. We show that any isometric action of the groups $\Gamma = St_n(R)$ when $R = \mathbb{F}_p\langle t_1, \dots, t_k \rangle$ ($n \geq 3$) on finite dimensional Hadamard manifold is finite.

Theorem 1.1. *Let $\Gamma = St_n(R)$ when $R = \mathbb{F}_p\langle t_1, \dots, t_k \rangle$, then any isometric group action of Γ on a finite dimensional Hadamard manifold X is finite, i.e. it factors through a finite group (in particular Γ has a fixed point in X .)*

Remark 1.2. Note that the (infinite dimensional) regular representation $\Gamma \rightarrow U(l^2(\Gamma))$ is a Γ isometric action which is not finite.

When the dimension of X is infinite it is not proper anymore and more delicate methods are needed. We restrict our treatment to pinched manifolds. These are manifolds whose sectional curvature is bounded from below as well. We show that this is enough to ensure that Γ has a fixed point in X , provided that $p \geq 5$.

Theorem 1.3. *Let Γ be as above with $p \geq 5$. If the sectional curvature of X is bounded from below (X can be of infinite dimension here) then Γ has a global fixed point in X .*

For the Steinberg groups defined over the ring $R = \mathbb{Z}\langle t_1, \dots, t_k \rangle$ such a theorem cannot be true. Being an unbounded subgroup in $SL_n(\mathbb{R})$, $SL_n(\mathbb{Z})$ is acting on the symmetric space associated with $SL_n(\mathbb{R})$ without a fixed point. Since $SL_n(\mathbb{Z})$ is a quotient of Γ this induces a fixed point free Γ isometric action. However following is true.

Theorem 1.4. *Let $\Gamma = St_n(R)$ ($n \geq 3$) when $R = \mathbb{Z}\langle t_1, \dots, t_k \rangle$. Suppose that X is a $CAT(0)$ space and that H is a group acting on X properly and co-compactly then any homomorphism $\phi : \Gamma \rightarrow H$ has a finite image.*

Remark 1.5. Recall that $SL_n(\mathbb{Z})$ is a non-uniform lattice in $SL_n(\mathbb{R})$. The theorem above gives a nice rigidity property. Namely, $SL_n(\mathbb{Z})$ can not be mapped onto co-compact lattices in $CAT(0)$ groups.

Remark 1.6. We point out that fixed point theorem for these groups acting on low dimensional $CAT(0)$ cell complexes was established by Farb (see [9].)

1.1. Ideas and Techniques. Results similar to that of Theorem 1.1 were obtained by Wang, followed by the work of Izeki and Nayatani (see [18], [11]) who showed that many lattices in semi-simple algebraic groups over p-adic field have fixed point property. As mentioned above Naor and Silberman also obtained fixed point property related to action of random groups on many convex spaces. In both cases the results were obtained by carrying some averaging process. This process yields some heat equations. Spectral gap ensures the process terminates with a fixed point. A key step is to obtain Poincare inequalities. Those are in general hard to obtain. The methods just described are inspired by Zuk's criteria used for proving property (T). Our techniques are also borrowed from methods used for proving property (T). We try to adopt the geometric approach.

The geometric approach towards proving property (T) was first introduced by Dymara and Januszkiewicz in [6], and then developed by Ershov, Jaikin and Kassabov [7] [12] [8].

The main idea is to examine angles between invariant spaces of finite (compact in the non-discrete case) subgroups generating Γ . Since these groups are finite, each of them has property (T) which means almost invariant vectors are "close" to invariant vectors. If on the other hand the angles between any two respective invariant vector spaces is "large enough" then the invariant vectors spaces of the finite groups are "far" from each other. The conclusion is that when no non-zero Γ invariant vectors exist almost invariant vectors are trivial and the group has property (T). In the case $\Gamma = \langle G_1, G_2, G_3 \rangle$ the meaning of "large enough" is that these angles' sum is greater than π (see [7] and [12]).

Our method is similar. We study the action of small subgroups of Γ and deduce from it about the large group. When proving fixed point property for Hadamard manifolds we seek for "fat" triangles. By saying "fat" we mean triangles in which, the sum of the angles is greater than π . We will present a triangle whose vertices are fixed by the finite groups and that the angles between any two sides of it is at least the angle between the invariant spaces. In our case, we look at triangle which is minimal in the sense that the sum of squares of lengths of its sides is minimal. As the sum of the angles in any CAT(0) space can't be larger than π we deduce that the triangle is a single point. Recently (and independently) Ershov and Jaikin adopted a similar method and proved a fixed point theorem regarding to isometric group actions of these groups on L_p spaces. Mimura [15] used different (purely algebraic) methods and proved fixed point properties related also to non commutative L_p spaces (provided that $n \geq 4$.)

1.2. Property FH. When the underlying space is a Hilbert space \mathcal{H} these ideas become very explicit. In this section we illustrate these ideas by giving an affine version of Kassabov's proof for the fact that these groups have property (T) (compare with Theorem 5.9 in [7] and Theorem 1.2 in [12]). We prove :

Theorem 1.7. *Let G be a group satisfying the following properties:*

- (1) $G = \langle G_1, G_2, G_3 \rangle$ where each pair G_i, G_j generates a finite group.
- (2) For any orthogonal representation (π, \mathcal{H}) of the groups $G_{i,j} = \langle G_i, G_j \rangle$, every $v \in \mathcal{H}$ satisfies the following property:

$$(1) \quad d_0^2(v) < 2(d_i^2(v) + d_j^2(v))$$

where $d_0(v)$ denotes the distance of v from $\mathcal{H}^{G_{i,j}}$, the (closed) space of $G_{i,j}$ invariant vectors, and $d_i(v)$ measure the distance between v and \mathcal{H}^{G_i} .

then G has property FH.

Remark 1.8. (1) It is readily verified that 1 is equivalent to having angles greater than $\pi/3$ between the corresponding subgroups as defined in the next section (see discussion in [12]). This together with 2.2 give the desire result regarding the Steinberg groups.

- (2) The fact that $G_{i,j}$ are finite ensures that $\mathcal{H}^{G_{i,j}}$ is not empty in any G isometric affine action.

As explained above we are interested in fat triangles. We will introduce one by minimizing the radius of the barycentric circle. Given an affine isometric G action, (ρ, \mathcal{H}) we define a function $f : \mathcal{H} \rightarrow \mathbb{R}$, by

$$x \mapsto d^2(x, \mathcal{H}^{G_{1,2}}) + d^2(x, \mathcal{H}^{G_{1,3}}) + d^2(x, \mathcal{H}^{G_{2,3}}).$$

Claim 1.9. Suppose that (ρ, \mathcal{H}) is an isometric affine action and f is the function defined above then f attains a minimum.

Proof. Indeed the affine map $x \mapsto ((x - \pi_{1,2}(x)), (x - \pi_{1,3}(x)), (x - \pi_{2,3}(x)))$ (with $\pi_{i,j}$ denoting the projection on $\mathcal{H}^{G_{i,j}}$) maps \mathcal{H} onto an affine subspace of $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$. A pre-image of the closest point to 0 in this subspace is minimal. \square

proof of Theorem 1.7. Suppose towards contradiction that (ρ, \mathcal{H}) is an affine isometric fixed point free G action. Let $q \in \mathcal{H}$ be a point minimizing f . For simplicity denote the projections of q on the fixed points spaces $\mathcal{H}^{G_{i,j}}$ by x, y, z . Note that since q is minimizing for f we can assume that it is the barycenter of $\{x, y, z\}$ this means $q = \frac{x+y+z}{3}$. Note that seen from each vertex, the restriction of the action to the corresponding subgroup is an orthogonal representation. By applying 1 three times and summing we obtain:

$$d^2(q, x) + d^2(q, y) + d^2(q, z) < 4(d_1^2(q) + d_2^2(q) + d_3^2(q)) \leq 4(d^2(q, [x, y]) + d^2(q, [x, z]) + d^2(q, [y, z]))$$

(the second inequality follows from the fact that the segment connecting two vertices is fixed by the intersection of the corresponding subgroups). However for a barycenter point in an Euclidean triangle this is impossible. Indeed it is well known that the barycenter lies on the intersection of the medians. The barycenter divides each median segment into two subsegments. The first connects the barycenter to the vertex and is twice as long as the second which connects the barycenter to the middle of the opposite side. In general the segment connecting the barycenter to the middle of the side is longer than the distance from the barycenter to that side. We have then that if q is the barycenter of any triangle $\{x, y, z\}$ then:

$$d^2(q, x) \geq 4d^2(q, [y, z]).$$

(and same for the other vertices.) This gives a contradiction and the statement is proved. \square

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2. PRELIMINARIES

2.1. Angles between Invariant Subspaces. Let H be a finite group acting on a Hadamard manifold X . Recall that Hadamard manifolds are complete simply connected non-positively curved Riemannian manifolds (possibly of infinite dimension). By a classical theorem of Cartan H fixes a point in X . Suppose that $x_0 \in X$ is fixed by H and that ξ is a geodesic ray issuing from x_0 , then ξ is mapped onto another ray also issuing from x_0 . The action

then reduces to a representation on the tangent space at x_0 , denoted by T_{x_0} . Furthermore as isometric maps preserves angles, this representation is actually orthogonal. This motivates the study of angles between invariant subspaces in orthogonal representations in the context of isometric actions on manifolds. Recall Kassabov's definition for angles between closed subspaces (see [12]):

Definition 2.1. Let V_1, V_2 be two closed subspaces in a Hilbert space. We define the angle between V_1 and V_2 to be the infimum over the angles between vectors $v_i \in V_i$ $i = 1, 2$ such that $v_i \perp V_1 \cap V_2$. i.e.

$$\angle(V_1, V_2) = \inf\{\angle(v_1, v_2) \mid v_i \in V_i \text{ and } v_i \perp V_1 \cap V_2\}$$

Note that this is equivalent to say that

$$\cos(\angle(V_1, V_2)) = \sup\left\{\frac{\langle v_1, v_2 \rangle}{\|v_1\| \|v_2\|} \mid v_i \in V_i \text{ and } v_i \perp V_1 \cap V_2\right\}$$

When V is a unitary representation of G we denote by V^G the (closed) subspace of invariant vectors in V . It is convenient then to define angle between subgroups:

Definition 2.2. Suppose $H = \langle G_1, G_2 \rangle$ the angle between G_1 and G_2 is defined as

$$\angle(G_1, G_2) = \inf\{\angle(V^{G_1}, V^{G_2}) \mid V \text{ is a unitary representation of } H\}$$

Remark 2.3. (1) A tangent space at a point $x \in X$ is real vector space. An isometric representation on a real vector space will be denoted Orthogonal while an isometric representation on a complex vector space will be denoted as Unitary.

(2) Given an orthogonal representation on a real vector space V , denote by $U = V^\mathbb{C}$ the complexification of V , $U = V \otimes \mathbb{C}$. Given subgroups G_1, G_2 and an orthogonal representation on a real vector space V , one can easily verify that $\angle(V^{G_1}, V^{G_2}) = \angle(U^{G_1}, U^{G_2})$. Therefore $\angle(G_1, G_2)$ forms a lower bound on angles between invariant subspaces in real vector space.

Remark 2.4. Recall that a representation of finite (compact) group can be decomposed as a direct sum of irreducible ones. Thus, when $G_{1,2}$ is finite the phrase "any unitary representation" in the Definition 2.2 is equivalent to "any irreducible unitary representation".

In the next section we are going to give a criterion for a group, generated by finite subgroups, to have fixed point property.

Theorem 2.5. Let $G = \langle G_1, G_2, G_3 \rangle$ where $G_{i,j} = \langle G_i, G_j \rangle$ $i, j = 1, 2, 3$ are finite groups. Suppose that G is acting isometrically on a Hadamard manifold X . If the sectional curvature of X is bounded from below (X can be of infinite dimension then) and there exist $\theta > \pi/3$ such that $\angle(G_i, G_j) \geq \theta$, $1 \leq i \neq j \leq 3$ then G has a global fixed point in X .

In order to apply Theorems 2.5 we need to study representation theory of finite subgroups of the Steinberg group.

2.2. The Steinberg Group Over a Unital Ring. Recall the definition of the Steinberg group over unital ring. Let R be any unital ring (in our case R will be $\mathbb{Z}\langle t_1, \dots, t_k \rangle$, or, $\mathbb{F}_p\langle t_1, \dots, t_k \rangle$). The Steinberg group over R of dimension n , $St_n(R)$ is defined to be the group generated by $x_{i,j}(r)$ where $r \in R$ and $1 \leq i \neq j \leq n$, subject to the relations:

- (1) $x_{i,j}(r_1)x_{i,j}(r_2) = x_{i,j}(r_1 + r_2)$
- (2) $[x_{i,j}(r_1), x_{j,k}(r_2)] = x_{i,k}(r_1r_2)$
- (3) $[x_{i,j}(r_1), x_{l,k}(r_2)] = 1$ when $j \neq l$.

Where $[x, y] = x^{-1}y^{-1}xy$.

Remark 2.6. (1) The map defined by $x_{i,j}(r) \mapsto e_{i,j}(r)$ ($e_{i,j}(r)$ is the elementary matrix with 1 on the diagonal, r in the (i, j) place and 0 elsewhere) can be extended to a surjection map: $\phi : St_n(R) \rightarrow EL_n(R)$ on the group generated by elementary matrices. If R is commutative there is a natural definition of determinant and $EL_n(R)$ is a subgroup of $SL_n(R)$ (the kernel of the determinant map).
 (2) This is related to Algebraic K-Theory. The quotient

$$SL_n(R)/EL_n(R)$$

is denoted as $SK_1(n, R)$. Further the kernel of ϕ is closely related to $K_2(R)$ (it is a subgroup of $K_2(R)$).

Example 2.7. When $R = \mathbb{Z}$ and $n \geq 3$ this becomes very explicit:

- (1) It is easy to verify that any matrix in $SL_n(\mathbb{Z})$ can be written as a product of elements of $EL_n(\mathbb{Z})$ hence $SK_1(n, \mathbb{Z})$ is trivial.
- (2) K_2 however is not trivial. For example

$$x = (x_{1,2}(1)x_{2,1}(-1)x_{1,2})^4$$

is an element of order 2 in the kernel of ϕ . It is true however that the kernel of ϕ has exactly two elements (for this see Theorem 10.1 of [14]). This gives an alternative description of $SL_n(\mathbb{Z})$ in terms of generators and relations.

Next we collect some basic facts regarding to $St_n(R)$ and its representation theory. Throughout assume $R = \mathbb{F}_p\langle t_1, \dots, t_k \rangle$ (similar results are true for $R = \mathbb{Z}_p\langle t_1, \dots, t_k \rangle$). The following claim is easily verified:

Claim 2.8. The group $St_n(R)$ is generated by the following subgroups:

- $G_1 = \langle x_{1,n-1}(1), x_{2,n-1}(1), \dots, x_{n-2,n-1}(1) \rangle \cong \mathbb{F}_p^{n-2}$.
- $G_2 = \langle x_{n-1,n}(1) \rangle \cong \mathbb{F}_p$.
- $G_3 = \langle x_{n,1}(a_0 + a_1t_1 + \dots a_k t_k) \dots x_{n,n-2}(a_0 + a_1t_1 + \dots a_k t_k) \rangle \cong (\mathbb{F}_p^{n-2})^k$.

Remark 2.9. It is easily verified that for any $1 \leq i \neq j \leq 3$ the groups $G_{i,j} = \langle G_i, G_j \rangle$ are finite.

We are interested in the angles between G_i and G_j .

Lemma 2.10. *Suppose $p \geq 5$, then there exist $\delta > 0$ such that for every irreducible unitary representation (π, V) of $\langle G_i, G_j \rangle$, $1 \leq i \neq j \leq 3$ the angle $\angle(V^{\pi(G_i)}, V^{\pi(G_j)}) > \pi/3 + \delta$. In particular $\angle(G_i, G_j) > \pi/3$*

(see Section 4.1 in [7]) The ideas behind the proof are illustrated in the next example:

Example 2.11. Assume $R = \mathbb{F}_p \langle t_1, \dots, t_k \rangle$ and that $n = 3$. The subgroup $\langle G_1, G_2 \rangle$ is isomorphic to the (order p^3) Heisenberg group H_p over \mathbb{F}_p . The Heisenberg group, H_p is generated by

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with the obvious identifications $G_1 \cong \tilde{G}_1 = \langle x \rangle$ and $G_2 \cong \tilde{G}_2 = \langle y \rangle$. Further note that H_p can be decomposed as a semi direct product: $H_p = \tilde{G}_1 \ltimes A$ where $A \cong \mathbb{F}_p^2$ is the abelian group

generated by z and y and $\tilde{G}_1 \cong \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_p \right\}$ (with this identification, the action

of \tilde{G}_1 on A is via left matrix multiplication). A complete description of the irreducible representations of this group is given in [7]. Given a unitary irreducible representation of H_p , (π, V) its restriction to A decomposes as a direct sum of characters upon \tilde{G}_1 acts (identifying the dual of A with itself the action is by inverse transpose multiplication). Given a character χ its orbit may have either p elements or it is fixed by \tilde{G}_1 (since \tilde{G}_1 is of order p). In the former case one obtains a p dimensional space. In the latter case, the center of H_p (which is the group generated by z) is acting trivially. In this case $\pi(x)$ intertwines the action of A hence by Schur's lemma the restriction of π to \tilde{G}_1 is a character. The representation of H_p is then a character factoring through the abelianization of H_p , $H_p / \langle z \rangle$ (which is homomorphic to \mathbb{F}_p^2). So far we found p^2 representations of dimension 1 and $p - 1$ of dimension p by counting we observe that we found all. Let us describe the latter more detailed: let e_1, \dots, e_p be the natural basis of \mathbb{C}^p and let η be a non-trivial p 'th root of unity. Define

$$\pi(x)e_i = e_{i+1} \text{ (cyclic) and } \pi(y)e_i = \eta^{i-1}e_i.$$

In this case the spaces of invariant vectors are: $H^{\tilde{G}_1} = \mathbb{C}(e_1 + \dots + e_p)$ and $H^{\tilde{G}_2} = \mathbb{C}e_1$ and $\cos(\angle(H^{\tilde{G}_1}, H^{\tilde{G}_2})) = \frac{1}{\sqrt{p}}$.

2.3. Ultra-Products. Next we recall the construction of ultraproducts of Hadamard manifolds. Limits of metric spaces can be a power full tool. In our case we will refine the metric in a given manifold. We will assume that the group is acting fixed point freely and use this assumption in order to construct a sequence of marked manifolds that become more and more flat. By taking a limit we obtain a Hilbert space upon which the group is acting without a fixed point.

In general, a sequence of metric space does not necessarily has a convergence subsequence. A nice way to overcome this problem is by passing to ultralimits. A more complete description of ultra limits of metric spaces can be found in chapter I.5 in [4].

Let (X_n, x_n) be a sequence of marked Hadamard manifolds. Fix a non-principal ultra filter \mathcal{U} on \mathbb{N} . The ultra-product of (X_n, x_n) with respect to \mathcal{U} , denoted by $(X_n, x_n)_{\mathcal{U}}$ is the quotient:

$$(X_n, x_n)_{\mathcal{U}} = \left(\prod_n (X_n, x_n) \right)_{\infty} / \mathcal{N}$$

where

$$\left(\prod_n (X_n, x_n) \right)_{\infty} = \{(y_n) | y_n \in X_n, \sup_n d(x_n, y_n) < \infty\}$$

and \mathcal{N} is an equivalent relation identifying sequences of zero distance:

$$\mathcal{N} = \{(y, z) \in \left(\prod_n (X_n, x_n) \right)_{\infty}^2 \mid \lim_{\mathcal{U}} d(y_n, z_n) = 0\}$$

Suppose that $\alpha_n : G \rightarrow \text{Isom}(X_n)$ are group actions on X_n . If for every group element $g \in G$, and every $y = (y_n) \in (X_n, x_n)_{\mathcal{U}}$, the sequence $d(\alpha_n(g)y_n, x_n)$ is bounded, (actually it is enough to assume this for $\alpha_n(g)x_n$) the following formula is well defined and produces an isometric action on the limit space.

$$(2) \quad \alpha(g)(y) = (\alpha_n(g)y_n)$$

Example 2.12. I. An ultralimit of geodesic complete spaces is also geodesic complete.

An ultralimit of complete spaces is also complete (see [4].)

II. Ultra limit of CAT(0) spaces is also CAT(0) space. Indeed CAT(0) spaces are characterized by the property that for every triple of points x, y, z the following inequality holds:

$$d^2(x, m(y, z)) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z)$$

(where $m(y, z)$ is the midpoint between y and z .) Note that in inner product this is an equality. Moreover, complete geodesic complete, spaces for which this is equality are Hilbert spaces. This motivates the following example.

III. Suppose that X is an infinite dimensional Hadamard manifold whose sectional curvature is bounded from below, and that $\{x_n\}$ is any sequence in X . Suppose further that $\{\lambda_n\}$ is a sequence with $\lim_{n \rightarrow \infty} \lambda_n = \infty$ then the ultralimit of $(\lambda_n X, x_n)$ is a Hilbert space (where $\lambda_n X$ is the space X whose metric d is multiplied by λ_n .)

3. SUBSPACE ARRANGEMENTS AND FIXED POINT PROPERTY

We now begin with some useful facts to be used in the proof of Theorem 2.5. Throughout this section we assume that G is a group generated by finite groups, $G = \langle G_1, G_2, G_3 \rangle$. We further assume that any pair G_i, G_j generates a finite group. The main idea is to find "fat"

triangles whose vertices are fixed by the action restricted to $G_{i,j} = \langle G_i, G_j \rangle$. By assumption $G_{i,j}$ are finite. Hence by Cartan's theorem (see II.2.7 in [4]) they have fixed points.

Suppose that H is a finite group acting on a Hadamard manifold X . We denote by X^H the set of H fixed point in X . Note that when X is a Riemmanian manifold X^H is a closed submanifold. When X is a Riemmanian manifold and $x \in X$ is fixed by a group H we can treat the action of H as an orthogonal representation on the tangent space T_x . We wish to understand triangles whose vertices lie in $X^{G_{i,j}}$. We will do this in several steps. Recall that if $\{X_i\}_{i \in I}$ is a family of complete are complete CAT(0) spaces, their product $\prod_{i \in I} X_i$ with the L_2 metric is also a complete CAT(0) space . Let

$$T = X^{G_{1,2}} \times X^{G_{1,3}} \times X^{G_{2,3}},$$

and define

$$f : T \rightarrow \mathbb{R}^+, (x, y, z) \mapsto d^2(x, y) + d^2(z, y) + d^2(z, x)$$

for $x \in X^{G_{1,2}}, y \in X^{G_{1,3}}, z \in X^{G_{2,3}}$.

Remark 3.1. One can define also f^1 as $f^1(x, y, z) = d(x, y) + d(z, y) + d(z, x)$. Note that $f = 0$ iff $f^1 = 0$ and also $\inf f = 0$ iff $\inf f^1 = 0$. The advantage of defining f the way we did is that if we have $f \rightarrow \infty$ then f has unique minimum while f^1 has a minimum which is not necessarily unique. On the other hand calculations with f^1 are often easier.

We claim that a minimal triangle is "fat" i.e. the sum of its angles is grater than π . This will play a significant roll in the proof of Theorem 2.5 as the sum of angles in a triangle in CAT(0) space can't be greater than π . This should follow from our assumption on the angles between invariant subspaces in orthogonal representations. Indeed since we have fixed points, the restrictions of the action to the finite subgroups are orthogonal representations. This suggests that the angles between invariant submanifolds should also have sum which is greater than π . The problem is that our definition of angles "mod out" the intersection between the invariant subspaces. Geodesic path combining say the vertex x to y however, does not necessarily have derivatives perpendicular to $T_x^{(G_1, G_2)}$. The next claim deals with this matter:

Claim 3.2. Let $x \in X^{(G_i, G_j)}$ be a vertex in a minimal triangle as above, and let $c_1(t) \subset X^{G_i}, c_2(t) \subset X^{G_j}$ be the geodesic paths issuing form x to y, z respectively then $\angle(c_1'(0), c_2'(0)) \geq \angle(T_x^{G_i}, T_x^{G_j})$

Proof. Suppose that $\angle(c_1'(0), c_2'(0)) < \angle(T_x^{G_i}, T_x^{G_j})$. For convenience denote $V = c_1'(0)$ and $W = c_2'(0)$, also write $V = V_0 + V^\perp$ where $V_0 = P_{i,j}V$ is the orthogonal projection of V on $T_x^{(G_i, G_j)}$ (use the same notation for W). We will show that for some $x' \in X^{(G_i, G_j)}$ we get $f(x', y, z) < f(x, y, z)$. Since $\angle(V, W) < \angle(T_x^{G_i}, T_x^{G_j})$ we have that $\langle V_0, W_0 \rangle_{T_x} > 0$. This means that the angle between V and W_0 in T_x is acute. Now denote by $w = w(t)$ the exponent of W_0 in $X^{(G_i, G_j)}$. Since $\angle(w(t), c_1) < \pi/2$ and $\angle(w(t), c_2) < \pi/2$ we have that for some (every) t_0 small enough there exist $y' \in c_1$ and $z' \in c_2$ for which in the comparison triangles $\bar{\Delta}(\bar{x}, \overline{w(t_0)}, \bar{y}')$, and $\bar{\Delta}(\bar{x}, \overline{w(t_0)}, \bar{z}')$, the angles at \bar{x} will also be smaller

than $\pi/2$. This together with the CAT(0) inequality, would imply that for some (any close enough) point $p \in [x, w(t_0)]$ we would have $d(p, z') < d(x, z')$ and also $d(p, y') < d(x, y')$ and by the triangle inequality, $d(p, z) < d(x, z)$ and also $d(p, y) < d(x, y)$ hence also $f(p, y, z) < f(x, y, z)$. \square

Remark 3.3. One can use the claim above and prove a finite dimension version of 2.5. More precisely, one can prove that whenever such a group acts on a finite dimensional Hadamard manifold X fixed point freely, it must fix point in infinity. Indeed the function f defined above is convex hence if $f \rightarrow \infty$ as $x \rightarrow \infty$, it has a minimum (see for example [10]). That minimum is by the claim above a fixed point. On the other hand if f does not tend to infinity as x does, then by compactness of \tilde{X} it has a fixed point in infinity.

The proof of Theorem 2.5 demands a quantitative version of Claim 3.2. We will need to show that if f is bounded away from zero then triangles are "fat" even if they are not minimal but close enough to the infimum. More precisely:

Lemma 3.4. *Suppose that $\inf f = L > 0$ and that for every i, j the angle $\angle(G_i, G_j) \geq \theta > \pi/3$ then there exist $\epsilon > 0$ for which every triangle $(x, y, z) \in T$ with $f(x, y, z) < L + \epsilon$ has angles $> \pi/3$*

Proof. Fix $\epsilon > 0$ and suppose that the triangle (x, y, z) has $f(x, y, z) < L + \epsilon$. Assume towards contradiction that the angle at say x is smaller than $\pi/3$. Observe that there exist C (independent of ϵ) with $d(x, y), d(x, z) < C$. Observe further that assuming existence $c > 0$ (also independent of ϵ) with $d(x, y), d(x, z) > c$ doesn't cause any loss in generality. Indeed if $[x, z]$ is very small then by triangle inequality and the fact that $f(x, y, z) > L$ we have that $[y, z]$ is about the length of $[x, z]$ and therefore the angle at z is smaller than $\pi/3$ so we can get contradiction there.

Claim 3.5. There exist $W_0 \in T_x^{G_i, j}$ and $\alpha < \pi/2$ (independent of ϵ) such that $\angle([x, y], \exp(W_0)) < \alpha$ and $\angle([x, z], \exp(W_0)) < \alpha$.

Proof. Let $c_1(t) \subset X^{G_i}, c_2(t) \subset X^{G_j}$ be the geodesic paths issuing from x to y, z respectively. As above, also denote $V = c_1'(0)$ and $W = c_2'(0)$, and write $V = V_0 + V^\perp$ where $V_0 = P_{i, j}V$ is the orthogonal projection of V on $T_x^{(G_i, G_j)}$ (use the same notation for W)

By assumption we have

$$\frac{\langle V, W \rangle}{\|V\| \|W\|} = \frac{\langle V_0, W_0 \rangle}{\|V\| \|W\|} + \frac{\langle V_1, W_1 \rangle}{\|V\| \|W\|} > \frac{1}{2}$$

By our assumption on the angles between G_i and G_j , we have $\delta > 0$ for which:

$$\frac{\langle V_1, W_1 \rangle}{\|V\| \|W\|} < \frac{\langle V_1, W_1 \rangle}{\|V_1\| \|W_1\|} < \frac{1}{2} - \delta,$$

hence

$$\frac{\langle V_0, W_0 \rangle}{\|V\| \|W\|} > \delta$$

which by Cauchy Schwartz inequality implies:

$$\|V_0\| \|W_0\| > \delta \|V\| \|W\|.$$

Denote $w = w(t) = \exp(W_0) \subset X^{G_{i,j}}$. It follows then that the angle between $[x, y]$ and w as well as the angle between $[x, z]$ and w are bounded from above by $\alpha < \pi/2$. Indeed:

$$\frac{\langle W_0, W \rangle}{\|W_0\| \|W\|} = \frac{\langle W_0, W_0 \rangle}{\|W_0\| \|W\|} = \frac{\|W_0\|}{\|W\|} > \delta$$

and similarly

$$\frac{\langle W_0, V \rangle}{\|W_0\| \|V\|} > \delta.$$

□

Let then w be as in the claim and denote by x' and x'' the nearest point projections of y and z on w respectively. Without any loss in generality we assume that $d(x, x') < d(x, x'')$. We want to study the triangle (x', y, z) to get a contradiction. First observe that x' is closer to both y and z than x . We argue that for ϵ small enough we will get that $d(x, y) < c/2$ this will give us the desired contradiction. Indeed on the one hand we have for the comparison triangle $\bar{\Delta}(\bar{x}, \bar{x}', \bar{y})$, that the angle at x' is $> \pi/2$. Thus

$$d^2(x', x) \leq d^2(x, y) - d^2(x', y).$$

While on the other hand the triangle (x, y, z) has $f(x, y, z) < L + \epsilon$. Hence $f(x', y, z) - f(x, y, z) < \epsilon$ and in particular:

$$d^2(x, y) - d^2(x', y) < \epsilon.$$

Combining the two we see that $d^2(x, x') \leq \epsilon$.

On the other hand as the angle between $[x, y]$ and $[x, x']$ is bounded from above and the sectional curvature of X is bounded from below, $d(x', y)$ tends to zero as $d(x, x')$ does.

□

Our goal now is to prove Theorem 2.5. The proof however needs some extra preparation. In the proof we will use the same function f defined above. If we knew that f has a minimizing triangle we would apply Claim 3.2. Our goal then, is to show that indeed f attains a minimum

Assume then that f does not have a minimum. We will show that f is bounded away from zero. This will give us contradiction since then by Lemma 3.4 there exist a triangle with angles greater than $\pi/3$. To this end we define an auxiliary function h as follows:

Let $x \in X$ be any point and S be a closed submanifold of X . We denote by $\pi_S(x)$ the closest point projection of x on S . Now define:

$$h : X \rightarrow \mathbb{R}, \quad x \mapsto d(x, \pi_{X^{G_1, G_2}}(x))^2 + d(x, \pi_{X^{G_1, G_3}}(x))^2 + d(x, \pi_{X^{G_3, G_2}}(x))^2$$

One can easily observe that both f and h have zero infimum together namely:

Claim 3.6. $\inf_{x \in X} h(x) = 0$ iff $\inf_{\Delta \in T} f(\Delta) = 0$.

Proof. Indeed the if part follows from the triangle inequality. For the only if part consider the circumcenter of small triangle apply and the CAT(0) inequality. \square

Let $K < G$ be a compact (finite) symmetric generating set of G and x any point in X . Recall the definition of $diam(K \cdot x)$

Definition 3.7. $diam(K \cdot x) = \max_{k \in K} d(x, k \cdot x)$

The main step in proving that f is bounded away from zero is to construct a limit space upon G acts fixed point freely. In order to ensure absence of a fixed point we will need to bound the diameter of points which are closed to our base points. The next easy claim will help us in this task. It will enable us to replace "bad" points with "good" ones.

Claim 3.8. Let $K_1 = \langle G_2, G_3 \rangle$, $K_2 = \langle G_1, G_3 \rangle$, $K_3 = \langle G_1, G_2 \rangle$, and $K = K_1 \cup K_2 \cup K_3$. Suppose that $diam(K \cdot y) \leq \frac{1}{5}diam(K \cdot x)$ then $h(y) \leq \frac{1}{2}h(x)$

Proof. Suppose that $diam(K \cdot x) = d(x, gx)$. Without loss of generality we can assume that $g \in \langle G_1, G_2 \rangle = K_3$. Then by triangle inequality

$$d(x, \pi_{X\langle G_1, G_2 \rangle}) + d(gx, \pi_{X\langle G_1, G_2 \rangle}) \geq d(x, gx)$$

The action is by isometries and $\pi_{X\langle G_1, G_2 \rangle}$ is fixed by G hence this reads:

$$d(x, \pi_{X\langle G_1, G_2 \rangle}) \geq \frac{1}{2}d(x, gx) = \frac{1}{2}diam(K \cdot x).$$

In particular

$$(3) \quad h(x) \geq \frac{1}{4}diam(K \cdot x)^2.$$

Let y be a point in X with $diam(K \cdot y) \leq \frac{1}{5}diam(K \cdot x)$. Let c_i be the circumcenter of $conv(K_i \cdot y)$, i.e. c_i is the unique point minimizing the radius of ball containing the convex hull of $K_i \cdot y$. Then on the one hand (by definition of circumcenter)

$$d(y, c_i) \leq diam(K_i \cdot y) \leq diam(K \cdot y)$$

On the other hand c_i is K_i fixed hence

$$d(y, c_i) \geq d(y, \pi_{XK_i})$$

hence

$$h(y) \leq 3diam(K \cdot y)^2 \leq \frac{3}{25}diam(K \cdot x)^2 < \frac{1}{8}diam(K \cdot x)^2 \leq \frac{1}{2}h(x)$$

\square

We turn now to prove Theorem 2.5

Proof of Theorem 2.5. Let X be (possibly infinite dimensional) Hadamard manifold whose sectional curvature is bounded below by κ . Suppose that G is acting isometrically on X . Let f be defined as above. If f has a minimizing triangle then by Claim 3.2 this triangle is actually a point which is fixed by G . Suppose then towards contradiction that f does

not have a minimum. We will show that f is bounded away from zero. By Claim 3.6 it is enough to prove that h is bounded away from zero.

To this end we apply a limit process (compare with Lemma 3.1 in [2]).

Lemma 3.9. *Let X be a Hadamard manifold (possibly of infinite dimension) whose sectional curvature is bounded from below by κ , and suppose that G is acting on X fixed point freely, then $\inf_{x \in X} h(x) > 0$.*

Proof. Assume by contradiction that $h(z_n) \leq \frac{1}{2^n}$ for some sequence $z_n \in X$. Continue along the following steps:

I. Our first step is to construct out of it another sequence, having diameter bounded from below for nearby points, yet having vanishing of h .

Claim 3.10. There exist a sequence (x_n, k_n) (where $x_n \in X$ and $k_n \in \mathbb{N}$) with $h(x_n) \leq \frac{1}{2^{n+k_n}}$ and $\text{diam}(K \cdot y) \geq \frac{1}{5} \text{diam}(K \cdot x_n)$ for every $y \in B(x_n, \frac{1}{(n+k_n)^2})$.

Proof. Fix n and start with z_n . By the way we chose it $h(z_n) < \frac{1}{2^n}$. If however it happens that $\text{diam}(K \cdot y) < \frac{1}{5} \text{diam}(K \cdot x)$ for some $y \in B(z_n, \frac{1}{n^2})$, then by Claim 3.8 also $h(y) < \frac{1}{2} h(z_n) < \frac{1}{2^{n+1}}$. Denote $y_1^n = y$. If again it happens that $\text{diam}(K \cdot y) \leq \frac{1}{5} \text{diam}(K \cdot y_1^n)$ for some $y \in B(y_1^n, \frac{1}{(n+1)^2})$, then by Claim 3.8 we have again that also $h(y) < \frac{1}{2} h(y_1^n) < \frac{1}{2^{n+2}}$. Continue with this process obtaining a sequence y_k^n with $h(y_k^n) < \frac{1}{2^{n+k}}$.

Claim 3.11. This process has to terminate after finitely many times with $y_{k_n}^n$ which we denote by x_n .

Proof. Indeed otherwise $\{y_k^n\}_{k=1}^\infty$ is Cauchy sequence since $d(y_k^n, y_{k+1}^n) < \frac{1}{(n+k)^2}$. Since X is complete it has a limit which has to be a G fixed point. □

By construction x_n is the desired sequence. □

II. In the second step we construct a limit space. Let $X_n = \frac{1}{\text{diam}(K \cdot x_n)} X$ denote the Hadamard manifolds X with new metric $d_n = \frac{1}{\text{diam}(K \cdot x_n)} d$. The pointed spaces (X_n, x_n) has the following nice properties:

- (1) The sectional curvature of X_n is bounded from below by $\frac{\kappa}{\text{diam}(K \cdot x_n)}$.
- (2) The action of G induces an isometric action on X_n . In order to distinguish between the diameter of a point in X and the diameter in X_n we denote $\text{Diam}_n(K \cdot x) = \max_{k \in K} d_n(x, k \cdot x) = \frac{\text{diam}(K \cdot x)}{\text{diam}(K \cdot x_n)}$. By definition $\text{Diam}_n(K \cdot x_n) = 1$. Moreover for any sequence $y_n \in X_n$ for which $d_n(y_n, x_n)$ is bounded by some $L > 0$, $\text{Diam}_n(K \cdot y_n) \leq 2L + 1$.
- (3) On the other hand for every such y_n , $\text{Diam}_n(K \cdot y_n) \geq \frac{1}{5}$ for every n large enough (this follows from 3).

Fix a non principal ultra filter \mathcal{U} and let \mathcal{H} be the ultra product of the pointed spaces (X_n, x_n) . Then \mathcal{H} is a Hilbert space (see 2.12). Property 2 allows us to use 2 in order to define an isometric action on \mathcal{H} . This action is fixed point free by 3.

However it follows from Theorem 5.9 in [7] as well as Theorem 1.2 in [12], that G has property (T). By Delorme's Theorem G then has also property FH (see for example Theorem 2.12.4 in [3] and Theoreme V.1 in [5] or the direct proof we gave 1.7) hence we reached contradiction. \square

Since f is bounded away from zero there is a triangle Δ_0 whose angles are all greater than $\pi/3$ (by 3.4). This triangle then has to be a point fixed by G . \square

We can now prove 1.3.

Proof of Theorem 1.3. Theorem 1.3 follows from Theorem 2.5 combined with Lemma 2.2. \square

We turn now to prove Theorem 1.1. The proof relies on the well known fact that abelian groups that act on finite dimensional Hadamard manifolds without fixing any point must have element of infinite order. This fact follows from the fact that the fixed points set of any element is a complete Hadamard submanifold hence one can argue by induction.

Proof of Theorem 1.1. Suppose that α is an isometric Γ action. For fixed $1 \leq i \neq j \leq 3$, denote the abelian subgroup (isomorphic to the additive group of R),

$$H_{i,j} = \{x_{i,j}(r) \text{ s.t. } r \in R\} \cong R.$$

Then $H_{i,j}$ is an abelian group whose elements are of finite order, hence the restriction of α to $H_{i,j}$ fixes a point $x \in X$. Suppose then that $x \in X$ is fixed by $H_{i,j}$. Since the action is by isometries, the image of a point y is determined by the image of the geodesic segment $[x, y]$. The latter is determined by a finite dimension orthogonal representation on T_x which we denote by $\rho_{i,j}$. Then $\rho_{i,j}$ is a direct sum of one dimensional representations. Write:

$$\rho_{i,j} = \bigoplus_{k=1}^m \chi_k$$

(with $\chi_k \in \hat{R}$ characters on R and $m = \dim X$). Observe that as R is a direct sum of finite groups (namely copies \mathbb{F}_p), its dual \hat{R} is isomorphic then to the product $\prod_{n \in \mathbb{N}} \mathbb{F}_p$.

Claim 3.12. Let $A_k = \ker \chi_k$. Then (seen as a subgroup of R) $A_k < R$ is subgroup of finite index.

Proof. Indeed as the range of χ_k has p elements the kernel is of index p . \square

Corollary 3.13. Let $U_{i,j} < R = \{r \in R | x_{i,j}(r) \in \ker \rho_{i,j}\} = \bigcap_{k=1}^m A_k$. Further let $U = \bigcap_{1 \leq i \neq j \leq 3} U_{i,j}$ then U is a finite index two sided ideal in R .

Proof. Note first that for every i, j , if $u \in U$ then by definition $x_{i,j}(u)$ acts trivially. Now as it's a finite intersection of finite index subgroups it is also finite index and it is closed under addition. Suppose further that $u \in U$, take any $r \in R$ by the defining relations of the Steinberg group we obtain that $[x_{i,j}(r), x_{j,k}(u)] = x_{i,k}(ru)$. As $u \in U$ acts trivially the left hand side is also in the kernel, hence U is closed under left multiplication by elements in R . Observe that $[x_{i,j}(u), x_{j,k}(r)] = x_{i,k}(ur)$. It follows that U is a two sided ideal. \square

Next we adopt Milnor's notation (used in [14]). We denote by $St_n(U)$ the normal closer of the group generated by elements of the form $X_{i,j}(u)$, $u \in U$. This group is generated by elements of the form $sx_{i,j}(u)s^{-1}$ (with $s \in St_n(\mathbb{F}_p\langle t_1, \dots, t_k \rangle)$). It follows that $St_n(U) \triangleleft St_n(R)$ is in the kernel of α . By Lemma 6.1 in [14] we obtain a short exact sequence

$$1 \rightarrow St_n(U) \rightarrow St_n(R) \rightarrow St_n(R/U) \rightarrow 1.$$

On the other hand, Kassabov and Sapir showed that when R/U is finite, $St_n(R/U)$ is finite also (Lemma 17 in [13]). This proves that the kernel of α is of finite index and finishes the proof. \square

3.1. The Torsion Free Case. We now turn to deal with the groups $EL_n(R)$ when $R = \mathbb{Z}\langle t_1, \dots, t_k \rangle$. As above our results will be slightly more general since we work with $St_n(R)$ instead. Note that although (similarly to the case $R = \mathbb{F}_p\langle t_1, \dots, t_k \rangle$) these groups are still generated by groups of the form

- $G_i = \{x_{i,i+1}(a)\}$, where $a \in \mathbb{Z}$, $1 \leq i \leq n-1$, and
- $G_n = \{x_{n,1}(a_0 + a_1t_1 + \dots + a_k t_k)\}$, where $a_i \in \mathbb{Z}$, $1 \leq i \leq k$.

these groups are infinite hence our method would fail in the first step. Indeed these groups can act by hyperbolic isometries without fixing any point at all. However much is known about co-compact proper actions of solvable groups on CAT(0) spaces. Our main tool in proving 1.4 will be the solvable subgroup theorem which we will describe next.. We begin by reminding the definition of a metrically proper action.

Definition 3.14. Let G be a discrete group acting on a CAT(0) space X by isometries. We say that the action is metrically proper if for every $x \in X$ there is $r > 0$ such that the set $\{g \in G \text{ s.t. } g.B(x, r) \cap B(x, r) \neq \emptyset\}$ is finite.

Note that this definition is in general more restrictive than the usual definition of proper actions, which regards to compact sets in X . Even though one can clearly see both definitions coincide in the case of proper spaces (see definition I.8.2 and the following remark in [4]). The solvable subgroup theorem states that if a group Γ acts metrically proper and co-compactly on CAT(0) space then any solvable subgroup $S < \Gamma$ is finitely generated and more important, it is virtually abelian. It is straightforward to deduce from it that non-uniform irreducible lattices of higher rank semi simple Lie groups that have no compact factors can not act metrically proper and co-compactly on CAT(0) spaces (see theorem

II.7.8 and the following remark in [4]. Thus Theorem 1.4 is a generalization of this. We can now prove Theorem 1.4.

Proof of Theorem 1.4. Let X be a CAT(0) space upon H acts properly and co-compactly. Suppose further that we have a group homomorphism: $\phi : \Gamma \rightarrow H$. Similarly to the case studied above we denote $G_{i,j} = \langle x_{i,i+1}(R), x_{j,j+1}(R) \rangle$. We study the image of the solvable (Heisenberg) group $G_{1,2}$. Observe that by simple calculation the derived subgroup $[G_{1,2}, G_{1,2}]$ is just the subgroup $E_{1,3}(R)$ of matrices with 1 on the diagonal, elements of R in the $(1, 3)$ position and 0 elsewhere. By the solvable subgroup theorem the image of $G_{1,2}$ is virtually abelian hence $\ker \phi \cap [G_{1,2}, G_{1,2}]$ is of finite index in $[G_{1,2}, G_{1,2}] = E_{1,3} \cong R$.

For fixed $1 \leq i \neq j \leq n$ denote $U_{i,j} = \ker \phi \cap x_{i,j}(R)$. We proceed in a similar manner to the end of the proof of Theorem 1.1. First we show that for any i, j we have $U_{i,j} = U_{1,3}$ (with the obvious abuse of notation). Indeed if $r \in U_{1,3}$ then since $[x_{1,3}(r), x_{3,k}(1)] = x_{1,k}(r)$ then $r \in U_{1,k}$ for any $k \neq 1$. But then $[x_{k,1}(1), x_{1,j}(r)] = x_{k,j}(r)$ applies that $r \in U_{k,j}$ for any $k, j \neq 1$. Finally $[x_{k,j}(r), x_{j,1}(1)] = x_{k,1}(r)$ gives that $r \in U_{j,1}$ (one gets the opposite inclusion similarly). The groups $U_{i,j}$ are independent of i, j so we denote them by U . Next we show that U (seen as a subring of R) is a finite index two sided ideal. Indeed by definition it is a finite index (additive) subgroup in R . Moreover if $r \in U$ and s is any element in R then $[x_{1,3}(r), x_{3,k}(s)] = x_{1,k}(rs) \in U$ therefore U is closed under right multiplication by elements of R and similarly it is also a left ideal.

We obtain again a short exact sequence

$$1 \rightarrow St_n(U) \rightarrow St_n(R) \rightarrow St_n(R/U) \rightarrow 1$$

and again use the fact that $St_n(R/U)$ is finite when R/U is finite to deduce that the image of ϕ is finite. \square

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